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STABILITY ESTIMATION OF A SOLUTION IN ONE INTERNAL PROBLEM FOR THE LAPLACE EQUATION

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ANNOTATION

Internal problems for Laplace equation are non-correct problems, that have important theoretical and applied values. This article provides an estimate for the analytical continuation of stability in solving one internal problem for the Laplace equation.

Keywords: Internal problems, boundary value problems, non-correct problems, harmonic, analytical continuation.

The present place of mathematics in life and in science is determined by the fact that it allows us to translate intuitive approaches to reality. It is known that many topical problems are formulated in terms of mathematics. For example, when studying physical fields, oscillations, problems of potential flow around bodies in a liquid and for prospecting for minerals, problems associated with the Laplace equation are solved. All these problems, depending on the process under consideration, are internal or boundary value problems for the Laplace equation. Internal problems for the Laplace equation are ill-posed problems that have important theoretical and applied values [1].

It is known that the data included in the equation and in additional conditions in the physical problem are determined experimentally. In this paper, one of such problems is considered. The results of the study allow constructing an algorithm for an approximate solution of this problem. Letbe

$$\Delta u = 0, (r,\varphi) \in G(R) \quad (1)$$

$$|u(r_k,\varphi_1| \le \varepsilon_1 \quad (2)$$

$$\left|\frac{\partial u}{\partial r}(r_k,\varphi_2)\right| \le \varepsilon_2 \quad (3)$$

Where $\Delta -$the Laplace operator in a polar coordinate system,

$|r_k,\varphi_1| \le \varepsilon_1 r_k = \frac{k}{m}, k = 1,2,...,m,$ m-given natural number, $\varphi_1,\varphi_2 \in [0;\pi]$, $G(R)$ -

circle with center at point (0; 0), radius R [2]. Let us consider the solution of this problem in the set of harmonic $G(R)$,continuous on $\bar{G}(R)$ functions bounded in modulus by a constant $M > 0$. We introduce the following notation:

$S = \sum_{k=1}^{\infty} k\left(\frac{\rho}{R}\right)^{k-1}, S_1 = \frac{1}{\theta}\sum_{k=1}^{\infty} k^{\sigma}\frac{1}{R^k}, S_2 = \frac{1}{\theta}\sum_{k=1}^{\infty} k^{\sigma-2}\frac{1}{R^k},$ Where ρ,θ,σ fixed numbers respectively from intervals $(4e;R),(0;1),[1;\infty)$.

THEOREM. Let the following conditions be satisfied in problem (1) - (3):

1. $\varepsilon_1, \varepsilon_2$ **and natural number m are related by the relations:**

$$\varepsilon_1 = M\frac{\rho(R+\rho)}{(R-\rho)(\rho-\rho_0)}\left(\frac{\rho_0}{\rho}\right)^m, \varepsilon_2 = \frac{4MS\rho(R+\rho)}{R(\rho-\rho_0)}\left(\frac{\rho_0}{\rho}\right)^m,$$

Where ρ_0 **- fixed number,** $1 \le \rho_0 \le \rho;$

2. $\varphi_1 u \varphi_2$ **such that** $\varphi_2 - \varphi_1 = p\pi$ **,p-an irrational number and** $\left|\sin m(\varphi_2 - \varphi_1)\right| \ge \frac{\theta}{m^\sigma};$

3. $r_k = \frac{k}{m}.k = 1,2,\ldots,m;$ **Then, to solve problem (1) - (3) in the circle** $G(\rho_0) = \rho_0 \in \left[1; -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{2\rho}{e}}\right]$ **there is an estimate**

$$\left|u(r;\varphi)\right| \le \left[1 + \frac{2e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}\right].\left[\varepsilon_1(1+S_1) + \varepsilon_2 S_2 \rho R\right] \qquad \textbf{(4)}$$

Evidence. The harmonic function in a circle is represented as

$$u(r;\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\varphi + b_k \sin k\varphi)\left(\frac{r}{R}\right)^k$$

Where $a_0 = \frac{1}{k}\int_{-\pi}^{\pi} u(r;\varphi)d\varphi, a_k = \frac{1}{k}\int_{-\pi}^{\pi} u(r;\varphi)\cos k\varphi d\varphi, b_k = \frac{1}{k}\int_{-\pi}^{\pi} u(r;\varphi)\sin k\varphi d\varphi, k = 1,2,\ldots$

From coefficient representations $a_0, a_k, b_k, k = 1,2,\ldots$ follows that

$|a_0| \le 2M, |a_k| a \le 2M, |b_k| \le 2M, k = 1,2,\ldots$ Consider the function

$f_1(z) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\varphi_1 + b_k \sin k\varphi_1)\left(\frac{z}{R}\right)^k$ which is an analytical continuation of the function $u(r;\varphi_1)$ of $[0:R)$ in a circle $G(R) = \{z : |z| < R\}$.

Function $f_1(z)$ in a circle $G(\rho)$ bounded in modulus $\left|f_1(z)\right| \le M\frac{R+\rho}{R-\rho}$.

And at points $z = r_k, r = 1,2,3,\ldots.m$ satisfies the conditions $\left|f_1(r_k)\right| \le \varepsilon_1, k = 1,2,\ldots m.$ Analytical function $f_1(z)$ can be written as $f_1(z) = P_{1,m-1}(z) + R_{1,m}(z)$ Where $P_{1,m-1}(z) = \sum_{k=0}^{m-1} f_{1,k}.z^k$ and $R_{1,m}(z) = \sum_{k=m}^{\infty} f_{1,k} z^k$ .By the Cauchy inequality

$\left|f_{1,k}\right| \le M\frac{R+\rho}{R-\rho}\frac{1`}{\rho^k}, k = 1,2,\ldots..$ Now $R_{1,m}(z) = \sum_{k=m}^{\infty} f_{1,k} z^k$ we estimate modulo in the circle

$$G(\rho_0)=\{z:|z|<\rho_0\}:|R_{1,m}(z)|\le\frac{M\rho(R+\rho)}{(R-\rho)(\rho-\rho_0)}\left(\frac{\rho_0}{\rho}\right)^m=\varepsilon_1. \qquad (5)$$

Using the Lagrange interpolation formula, the polynomial $P_{1,m-1}(z)$ can be written as [3]

$$P_{1,m-1}(z)=\sum_{k=1}^{m}P_{1,m-1}.(r_k)\prod_{\substack{q=1\\ q\neq k}}\frac{z-r_q}{r_k-r_q}.$$

From the inequality $|f_1(z)-P_{1,m-1}(z)|\le\varepsilon_1$ follows that $|P_{1,m-1}(z)|\le 2\varepsilon_1, k=1,2,....m.$ (6)

Consider the polynomial $L_{1,m-1}(z)=\sum_{k=1}^{m}\prod_{\substack{q=1\\ q\neq k}}\frac{z-r_q}{r_k-r_q}$ in a circle $G(\rho_0)=\{z:|z|<\rho_0\}$ .From number representation $r_k=\frac{k}{m}$ and from the following combination property $C_{m-1}^0+C_{m-1}^1+...+C_{m-1}^{m-2}+C_{m-1}^{m-1}=2^{m-1}$ it follows that the estimate

$$|L_{1,m-1}(z)|\le\frac{[2m(1+\rho_0)]^{m-1}}{(m-1)!}$$

AccordingtoStirling'sformula

$$m!=\sqrt{2\pi m}m^m e^{-m}(1+\omega_m),0\le\omega_m<e^{\frac{1}{4m}}-1,$$

The last inequality takes the form

$$|L_{1,m-1}(z)|\le\frac{e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}} \qquad (7)$$

Then it follows from (6) and (7) that

$$|P_{1,m-1}(z)|\le\frac{2\varepsilon_1 e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}$$

And so, for the function $f_1(z)$ in a circle $G(\rho_0)=\{z:|z|<\rho_0\}$ there is an estimate

$$|f_1(z)|\le\varepsilon_1\left\{1+\frac{2\varepsilon_1 e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}\right\}$$

From the last estimate by the Cauchy inequality, we obtain

$$|f_1^{(k)}(0)|\le\varepsilon_1\left\{1+\frac{2\varepsilon_1 e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}\right\}.\frac{1}{\rho_0^k},k=1,2,..... \qquad (8)$$

Now consider the function $f_2(z)$, defined by the formula

$$f_2(z)=\sum_{k=1}^{\infty}\frac{k}{R}(a_k\cos k\varphi_2+b_k\sin k\varphi_2)\left(\frac{z}{R}\right)^k$$

Which is an analytical continuation of the function

$\frac{\partial u}{\partial r}(r,\varphi_2) = f_2(r)$ of $[0;R)$ in a circle $G(R) = \{z : |z| < R\}$.Obviously, it satisfies the estimate, $|f_2(z)| \le \frac{4MS}{R}$ and at points $\{z = r_k\}$true $|f_2(r_k)| \le \varepsilon_2$. Using the above scheme, it is easy to prove that the following estimates are valid

$$|P_{1,m-1}(z)| \le 2\varepsilon_2, \text{ and } |R_{1,m}(z)| \le \varepsilon_2 \qquad (9)$$

For function $f_2(z)$ in a circle $G(\rho_0) = \{z : |z| < \rho_0\}$get

$$|f_2(z)| \le \varepsilon_2 + \frac{2\varepsilon_2 e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}\left(\frac{1}{\rho_0}\right)^k$$

Andtherefore

$$\left|f_2^{(k)}(0)\right| \le \varepsilon_2 + \frac{2\varepsilon_2 e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}\left(\frac{1}{\rho_0}\right)^k \qquad (10)$$

Then for the harmonic function

$$u(r;\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\frac{1}{k!\sin k(\varphi_2-\varphi_1)}\left\{f_1^{(k)}(0)\sin k(\varphi_2-\varphi\ ) + Rf_2^{(k-1)}(0)\sin k(\varphi-\varphi_1)\right\}\left(\frac{r}{R}\right)^k.$$

We have $|u(r;\varphi)| \le \left\{1 + \frac{2e[2e(1+\rho_0)]^{m-1}}{\sqrt{2\pi m}}\right\}\{\varepsilon_1(1+S_1) + \varepsilon_2 S_2 R\rho\}$

The theorem is proved.

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