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FIND ALL INTEGER SOLUTIONS OF THE EQUATION IN THE FORM $ax^2 + bxy + cy^2 = d, a, b, c, d \in \mathbb{Z}^+$

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ANNOTATION

In this work, the simplest methods of finding integer solutions of the equations under consideration $ax^2 + bxy + cy^2 = d$ are given and are carried out on a theoretical basis. Solutions of problems are given for the cases $D = b^2 - 4ac \geq 0, \sqrt{D} \in \mathbb{Z}^+$ and $D < 0$.

Key words. integer solution, natural solution, GCD (greatest common divisor), LCM (least common multiple), system of linear equations.

Let us be given an equation of the form $ax^2 + bxy + cy^2 = d$ (1). Let $D = b^2 - 4ac$ be included in this equation.

Case 1. Let $D \geq 0, \sqrt{D} \in \mathbb{Z}$. In this case, we divide the left-hand side of the equation by factors such as $t_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$ – rational numbers $a(x - t_1 y)(x - t_2 y)$. We write the given equation as follows:

$$(2ax - 2at_1 y)(2ax - 2at_2 y) = 4ad \Leftrightarrow (a_1 x + b_1 y)(a_2 x + b_2 y) = d_1$$

here $a_1 = 2a; b_1 = 2at_1; a_2 = 2a; b_2 = 2at_2; d_1 = 4ad$ –integer numbers.

We look at the equation

$$(a_1 x + b_1 y)(a_2 x + b_2 y) = d_1 \qquad (2)$$

here $a_1, b_1, a_2, b_2 \in \mathbb{Z}^+; x, y \in \mathbb{Z}$.

Corollary. The equations $ax - by = d$ and $ax + bu = d$ have a whole solution at the same time**.**

Theorem. If d_1 is not divided into $GCD(a_1, b_1)$ or $\mathrm{GCD}(a_2, b_2)$, then equation (2) has no solution in integer numbers.

Proof. Suppose that equation (2) has a integer solution. d_1 $\mathrm{EKUB}(a_1, b_1) = k$ do not divide by k. By the definition of GCD,

$$a_1 = ku, b_1 = kv, \mathrm{EKUB}(u, v) = 1$$

Hence we come to equation

$$k(ux + vy)(a_2 x + b_2 y) = d_1$$

This means that d_1 divided by $GCD(a_1, b_1) = k$. This contradiction proves the theorem.

If d_1 is divided into GCD(a_1, b_1) or $GCD(a_2, b_2)$, then we reduce equation (2).

Therefore, we can assume that $GCD(a_1, b_1) = GCD(a_2, b_2) = 1$.

We can equate $a_1x + b_1y$ and $a_2x + b_2y$ in equation (2) to the integers $d_1$in pairs. Suppose that d_1 is the product of two integers $d_1 = c_1c_2$. Then find the integer solutions of equation (2)

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

(3) is equally finding all the solutions of a visual system.

Example 1. Find all the integer numbers of equation $55x^2 - 12xy - 91y^2 = 59$.

Solution. Let's do some calculations and divide the right side of the equation by factors:

$$(5x - 7y)(11x + 13y) = 59.$$

The number 59 can be written as a product of two integers as follows:

$$59 = 1 \cdot 59 = 59 \cdot 1 = (-1) \cdot (-59) = (-59) \cdot (-1)$$

The result is a system of four linear equations:

$$\begin{cases} 5x - 7y = 1 \\ 11x + 13y = 59 \end{cases}; \begin{cases} 5x - 7y = 59 \\ 11x + 13y = 1 \end{cases}; \begin{cases} 5x - 7y = -1 \\ 11x + 13y = -59 \end{cases}; \begin{cases} 5x - 7y = -59 \\ 11x + 13y = -1 \end{cases}$$

We get integers from the solutions of these four equations.

Answer: $\begin{cases} x = 3 \\ y = 2 \end{cases}; \begin{cases} x = -3 \\ y = -2 \end{cases}.$

Case 2. Let $D = b^2 - 4ac < 0$ be in equation $ax^2 + bxy + cy^2 = d, a, b, c, d \in \mathbb{Z}$. In this case, we multiply both sides of the equation by $4a$ and going to square the left side,

$$(2ax + by)^2 + (4ac - b^2)y^2 = 4ad \qquad (4)$$

Since the left-hand side of equation (4) is negative, equation (4) has no solution when $ad < 0$. If $ad \geq 0$ then, the following inequalities are satisfied:

$$\begin{cases} (2ax + by)^2 \leq 4ad \\ (4ac - b^2)y^2 \leq 4ad \end{cases} \Rightarrow |y| \leq 2\sqrt{\frac{ad}{4ac - b^2}} \qquad (5)$$

We find (5) by selecting all the integer y. Let y_0 be this integer. Substituting the found y_0 into the second inequality, we obtain the inequality $(2ax + by)^2 \leq 4ad \Leftrightarrow -2\sqrt{ad} \leq 2ax + by_0 \leq 2\sqrt{ad} \Leftrightarrow$

$$\Leftrightarrow -2\sqrt{ad} - by_0 \leq 2ax \leq -by_0 + 2\sqrt{ad} \qquad (6)$$

and find the integer x from (6).

Example 2. Find the pair of integers $(x; y)$ that satisfies the equation:

$$x^2 - 2xy + 2y^2 = 9.$$

Solution. We write the equation as follows:

$$(x-y)^2 + y^2 = 9$$

The result is the following system of inequalities:

$$\begin{cases}(x-y)^2 \leq 9 \\ y^2 \leq 9\end{cases}$$

$y = -3, -2, -1, 0, 1, 2, 3$ are integers that satisfies the inequality $|y| \leq 3$.

We find them by putting in the inequality $|x-y| \leq 3$ and selecting the ones that satisfy the given equation:

$$\begin{cases}x=-3\\ y=-3\end{cases}; \begin{cases}x=\pm 2\pm\sqrt{5}\\ y=\pm 2\end{cases}; \begin{cases}x=\pm 1\pm\sqrt{8}\\ y=\pm 1\end{cases}; \begin{cases}x=\pm 3\\ y=0\end{cases}.$$

Answer: $\begin{cases}x=-3\\ y=-3\end{cases}; \begin{cases}x=-3\\ y=0\end{cases}; \begin{cases}x=3\\ y=0\end{cases}.$

Example 3. Find all natural solutions of the equation $2x^2 + 3xy + y^2 = 3^{2002}$.

Solution. Let's do some calculations and divide the right side of the equation by factor:

$$(x+y)(2x+y) = 3^{2002}.$$

From this equation we form the system of equation

$$x + y = 3^a, 2x + y = 3^b, a + b = 2002,$$

and from it we form following equations $x = 3^b - 3^a, y = 2 \cdot 3^a - 3^b$ (here $a > 0, b > 0$). It is obviously that $x \in \mathbb{N}$ va $y \in \mathbb{N}$. Hence, since $3^b > 3^a$ and $3^{a+1} > 2 \cdot 3^a > 3^b$, the given equation has no solution in natural numbers.

CONCLUSION

The given method of solving the equation in the form $ax^2 + bxy + cy^2 = d, a, b, c, d \in \mathbb{Z}$ in integers is recommended for teachers of specialized schools and academic lyceums.

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